Structural Stability and Goodwin's *A Growth Cycle*. A Survey.[1]

Roberto Veneziani

Department of Economics, London School of Economics and Political Sciences, Houghton Street, London WC2A 2AE

e-mail: R.Veneziani@lse.ac.uk

Abstract: In this paper, the methodological prescription to reject a structurally unstable model is critically analysed by considering Goodwin's *A Growth Cycle* [15, 16]. Moreover, a survey of the literature on Goodwin's model (GM) is provided, focusing on the issue of structural instability.

It is argued that, both theoretically and empirically, GM is a more satisfactory formalisation of distributive conflict than structurally stable models. An interpretation of GM based on its structural instability is proposed, arguing that GM isolates *in vitro* the main forces underlying distributive conflict, and should not be used to *describe* business cycles in a perturbed environment. Finally, an explanation of the empirical evidence is provided based on this interpretation of GM.

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"We should look for precision in each class of things just so far as the nature of the subject admits" Aristotle, *Nichomachean Ethics*, bk. I: ch.3, 1094b 25.

Introduction.

Although its origins can be traced back to Greek philosophers, the concept of *structural stability* has been formalised with the tools of modern topology only recently, in a seminal contribution by Andronov and Pontryagin [2].[2] Unlike *dynamic* stability, which relates to a property of a state or an orbit of a dynamic system, structural stability relates to a property of a dynamic system itself, and it incorporates the idea that the qualitative features of the phase portrait do not change in an essential way, if the system is slightly perturbed.

In the received view, structural stability represents a fundamental criterion in model building. The main argument underlying the methodological prescription to reject a structurally unstable model is similar to Samuelson's [32] famous argument against a *dynamically* unstable equilibrium. Namely, "it would be infinitely improbable to observe the existence of unstable structures because they are transient and unobservable." [45, p.51] Structural stability is considered a necessary condition for the observability and the predictability of scientific phenomena, while a structurally unstable model is deemed inherently unable to describe an empirical regularity and inadequate for predictive purposes.

However, the methodological prescription to reject a structurally unstable model has recently been put into question. For instance, in economics, "rational expectation macroeconomics has recently reversed this methodological principle"

[2] A thorough discussion of the *philosophical* concept of structural instability and its evolution in the philosophy of sciences is provided in Vercelli [45].

[45, p.43] by adopting models characterised by a saddle point whose behaviour is structurally unstable. More generally, in a path-breaking contribution Vercelli [45] has challenged the received view from a broad methodological perspective, showing that structural instability should not be sufficient to reject a model *a priori* and suggesting the need of a less dogmatic approach to model building.

Due to the scope of the philosophical issues raised, in this paper the methodological prescription to reject a structurally unstable model is not discussed from a general perspective. Instead, the received view on the inadequacy of structurally unstable models *in economics* is discussed by analysing the specific case of Goodwin's seminal paper *A Growth Cycle* [15, 16].

Goodwin's model (GM) represents one of the first and most elegant dynamic formalisations of Marx's theory of distributive conflict and, despite its stylised structure, a very original synthesis with more recent developments in economic theory (e.g. the relation between unemployment and wages is modelled by means of a Phillips' curve). Moreover, *A Growth Cycle* represents a seminal contribution in the use of nonlinear dynamic models (such as those developed in mathematical biology) in economics. Thus, GM has proved extremely fruitful in stimulating a vast literature aimed at analysing the positive and normative properties of the model. However, in GM the cyclical interaction between employment and the workers' share of income is modelled by a *structurally unstable* Lotka-Volterra dynamic system (LV) [26, 46]. In most of the literature this has been considered as the "main defect" [29, p.17], the "worst fault of the system" [11, p. 514], an "undesirable quality for economic models" [37, p.37]. And many of the extensions of GM have been aimed at avoiding structural instability, based on the view that if GM "is to provide a satisfactory model of economic cycles, changes must be made to its mathematical structure so that the governing differential equations exhibit stable limit cycle behaviour" [5, p.70].

The main aim of this paper is to refute this view. Given Vercelli's [45] analysis, structural instability is not considered sufficient to reject GM *a priori*, and a different methodological approach is adopted, based on the analysis of (a) the main characteristics of the theory that the model is meant to formalise and (b) the empirical features of the phenomenon investigated. It is argued that, from both points of view, GM represents a more satisfactory formalisation of Marx's theory of distributive conflict than the structurally stable extensions yielding limit cycles. In order to support this view, an interpretation of GM based on its structural instability is proposed, arguing that GM is meant to isolate *in vitro* the main forces underlying distributive conflict, and it might be inappropriate to use it to *describe* the data pattern in a perturbed environment.

Although an exhaustive survey of the vast literature on GM would go beyond the scope of a paper, the main arguments of this contribution are based on a critical review of the literature. Therefore this paper also represents an attempt at filling a gap in the literature by presenting a partial survey of the contributions on *A Growth Cycle* focused on the issue of structural instability.[3]

The structure of the paper is as follows. In *section 1*, the main features of GM are briefly recalled and the topological concept of structural stability is discussed. Then, the game-theoretic attempts at a micro-foundation of the

[3] To the best of my knowledge there is no exhaustive review of the literature on GM. For a limited survey, see [31].

structurally unstable specification of GM are critically reviewed. In *section 2,* GM and structurally stable extensions are compared as alternative formalisations of distributive conflict. Given the empirical evidence on Goodwin cycles, in *section 3* an alternative interpretation of GM based on its structural instability is provided. The conclusions are in *section 4,* while the formal analysis is in the *Appendix*.

Section 1. Goodwin's Model and Structural Instability.

Section 1.1. The Model.

The assumptions of the model can be summarised as follows:[4]

A1) fixed capital (k) - output (q) ratio, $s \equiv k/q$, so that $\dot{k}/k = \dot{q}/q$;

A2) Say's law holds and profits are completely reinvested (with no savings out of labour income)

$$s = i = \dot{k} = q - wl = q(1 - u)$$

where s denotes savings, i investment, w the real wage rate, l the employed and $u \equiv wl/q$ the share of labour in national income. Dividing by k and using A1

$$\dot{k}/k = q(1 - u)/k = (1 - u)/s$$

A3) exogenous (constant) growth rates of productivity, $a \equiv q/l$, and of population, n, so that $a = a_0 e^{\alpha t}$ and $n = n_0 e^{\beta t}$. Thus $\dot{q}/q = \dot{l}/l + \alpha$;

Let $v \equiv l/n$ be the rate of *employment*. After some algebra, from A1-A3:

$$(1)\quad \dot{v}/v = [1/s - (\alpha + \beta)] - u/s$$

A4) The real wage bargaining equation (Phillips curve) is

$$\dot{w}/w = -\gamma + \rho v$$

[4] For a thorough mathematical analysis of GM, see e.g. [7] and [21].

where g, $r > 0$. Hence, given that $u = w/a$,

$$(2)\quad \dot{u}/u = -(g + a) + rv$$

Actually, (1)-(2) form a LV dynamic system, a special case of the dynamic systems studied by Kolmogorov [24]:

$$(3)\quad \dot{v} = F(v, u)$$

$$(4)\quad \dot{u} = G(u, v)$$

where F and G are linear and the zero restrictions on the parameters yield a LV model displaying conservative oscillations around a neutrally stable equilibrium, the *centre* [40, 12]. Despite its simple structure, GM has many interesting features that have been extensively analysed in the literature (the possibility of modelling cycles and growth together, its non-linear structure, etc.). This paper focuses on the *structural instability* of (1)–(2), a feature that raises important methodological issues and has led many authors to question the opportunity to adopt the model.

Section 1.2. A formal definition of structural stability.

Let $x, y \in \hat{A}^n$ and let $\langle x, y\rangle$ be the inner product of x, y. Let $D^n = \{x \in \hat{A}^n | \, |x| \leq 1\}$, where $|x|$ is the Euclidean norm of x. Let ∂D^n be the boundary of D^n and $\partial D^n = \{x \in \hat{A}^n | \, |x| = 1\}$. Let C^1 be the set of bounded and continuous functions on $\hat{A}^n$. Let W be an open set in $\hat{A}^n$ and let $V(W)$ denote the set of all C^1 vector fields on W. *Structural Stability* can be defined formally as follows

Definition. (Hirsch and Smale [22, p.312-3]). Consider C^1 vector fields $f: W \rightarrow \hat{A}^n$ defined on some open set W containing D^n such that $\langle f(x), x\rangle < 0$ for each $x \in D^n$. f is ***structurally stable*** on D^n if there exists a neighbourhood $I \subset V(W)$ such that if

g: $W \rightarrow \hat{A}^n$ is in I, then flows of f and g are topologically equivalent on D^n. I.e. there exists a homeomorphism h: $D^n \rightarrow D^n$ such that for each $x \in D^n$,

$$h(\{f_t(x)|\ t \geq 0\}) = \{y_t(x)|\ t \geq 0\},$$

where $f_t(x)$ is the flow of f, y_t is the flow of g, and if x is not an equilibrium, h preserves the orientation of the trajectory.[5]

In other words the topological definition of structural stability incorporates the intuitive idea that the features of the phase portrait do not change in an essential way if the system is slightly perturbed, by requiring that "nearby" dynamic systems have the same qualitative dynamics.

Hirsch and Smale's [22] definition is adopted here since it is the most commonly used in the literature on GM (e.g. [40, 12]), however it is by no means the only way of formalising the concept of structural instability.[6] As it is clear from the *Definition*, "the concept of structural stability is based on three other fundamental concepts: (i) a notion of equivalence of systems; (ii) a topology of systems; (iii) a notion of perturbations which are possible *a priori*. All three notions are neither natural nor unequivocal." [6, p. 96] Thus, there is an inherent ambiguity in the definition of structural instability and the differences between alternative formalisations do not reflect mere definitional problems.[7] Actually, "the choice of one definition rather than another depends on opportunity

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[5] The orientation of a trajectory is the direction that points move along the curve as t increases.

[6] For a discussion of alternative definitions of structural stability, see [1, 45].

[7] For instance, the topological definitions of structural stability normally exclude perturbations that alter the order of the system. However, "the excluded case may be of the utmost scientific interest. This is the case, for example, of the so-called 'parasitic' parameters which play an important role in many fields of physics and engineering" [45, p.48].

considerations which are related in their turn to the mathematical nature of the problem, … but also to the characteristics of the problem to which the mathematical instruments are applied.” [44, p.178]

For instance, if a definition of *topological equivalence* is adopted, it is possible to argue that GM is structurally unstable with respect to linear perturbations of the coefficients of the reduced form (1)-(2). In fact the *centre* is a non-*hyperbolic* equilibrium, i.e. the Jacobian of the linearised system around the *centre* has eigenvalues with real part equal to zero, and therefore a small linear perturbation can make it into a *sink* or a *source*. However, it is not true that “*every* minimal alteration of the structure of GM would destroy its main characteristic” [29, p.33, italics added], i.e. the closed orbits. Cugno and Montrucchio [6, pp.96-9] prove that GM is structurally stable with respect, e.g., to perturbations of the Phillips curve. Furthermore, by using a definition of *observational*, or *practical*, *equivalence* they prove the structural stability of the model under more general perturbations [6, p.99].

In the case of GM, the inherent ambiguities in the formal definition of structural stability are also reflected in the difficulty to provide a fully satisfactory proof of the structural instability of GM. On the one hand, as shown by Flaschel [12], Velupillai’s [40] attempt at applying the *Structural Stability Theorem* [22, p.314],[8] which characterises structurally stable systems in the plane, is flawed,

[8] **Theorem.** *Structural stability* [22, p.314]. “Let $f: W \rightarrow \hat{A}^2$ be a C^1 vector field on an open set $W \supset D^2$ such that f points inward along the boundary ∂D^2 of D^2, i.e. $\langle f(x), x \rangle < 0$ if $x \in \partial D^2$. Then f is structurally stable on D^2 if and only if (a) the equilibria in D^2 are hyperbolic and (b) each closed orbit in D^2 is either a periodic attractor or a periodic repeller and (c) no trajectory in D^2 goes from saddle to saddle.” As shown by Flaschel [12], f does not point inward along ∂D^2.

since GM does not lie in the restricted set of vector fields considered in the Theorem. On the other hand, the alternative proofs provided in the literature, such as [7, 12, 39, 5], that aim to replace "Velupillai's purely formal and insufficient proof by simple economic reasoning" [12, p.63], are not entirely convincing.

In these papers, GM is shown to correspond to a *bifurcation* value of the parameters of a more general model in which GM is embedded, so that if the parameters are slightly perturbed, the dynamics of the system change radically.[9] However, this kind of proof does not seem conclusive, due to the lack of a well-defined methodology, and in particular of a clear prescription concerning the extensions allowed in the proof. Thus, in principle, by simply choosing the appropriate generalisation, the structural instability of virtually any model could be proved, especially if extensions altering the order of the system were included. As it is well-known, the "likelihood" of structural stability decreases as the order of the system increases so that "even the genericity of structural stability in the compact manifold of dimension 2 would be jeopardized" [45, p.48, fn.5] by perturbations that alter the dimension of the system.

These observations are not meant to deny the structural instability of GM but only to cast it in a different perspective. They suggest that, in general, "the query of whether or not a certain system (model) is structurally unstable cannot be answered with a simple yes or no" [44, p.179];[10] and, more importantly, they

[9] The concept of *bifurcation* is closely related to that of structural stability, since at a bifurcation point a system discontinously changes its type of motion. See, e.g., [19] for a discussion.

[10] Moreover, a system may be structurally stable with respect to infinitesimally small perturbations but not with respect to perturbations of a small but finite magnitude. Hence, it would be more suitable to talk about "a degree of structural instability" [45, p.48]. Since GM is structurally unstable with respect to a class of linear *infinitesimal* perturbations this issue is not discussed here.

should raise some doubts about an a-critical application of the methodological prescription to reject a structurally unstable model.

Section 1.3. Microfoundations and structural instability.

In GM, agents always behave in the same way and the economy fluctuates around the *centre* forever. However, assuming diminishing marginal utility of consumption, both classes would prefer orbits characterised by smaller amplitude, the *centre* being the preferred orbit (with null amplitude). Hence, both classes have an incentive to modify their behaviour, e.g., by signing a mutually beneficial social contract allowing them to reach the *centre* immediately, and therefore GM might be subject to Lucas' famous *critique* [27].

This problem and the lack of a description of the agents' behaviour at a micro-level in GM have led some authors to a game-theoretical analysis of the interaction between the two classes. A thorough discussion of the methodological issue of microfoundations goes beyond the scope of this paper. However, if microfoundations were provided to the structurally unstable specification of GM, it would be possible to adopt an argument similar to that used by neo-classical economists to focus on *unstable* saddle-path equilibria in rational expectation models of the Ramsey-Cass-Koopmans type. It might be argued that structural instability would not be a major problem of GM, since the zero restrictions yielding the LV specification, would be *chosen* by the agents.[11]

[11] Although, e.g., stochastic shocks and measurement errors might still be relevant. For a thorough discussion of this argument, see [45, chp.6].

Balducci, Candela and Ricci (BCR) attempt to establish "conditions of robustness for Goodwin dynamic properties" [4, p.50] by differential games' tools. In their model fully unionised workers choose the intensity of their wage claims (the slope of the Phillips curve), while fully organised capitalists choose the percentage of their profits to be invested (thus A2 does not hold). Both classes minimise a quadratic loss function that depends on the difference between actual and desired parameters, subject to (1)-(2) describing the state dynamics. BCR's main result is that GM coincides with the equilibrium of a non-cooperative Nash game in which agents' optimal strategies correspond to Goodwin's parameters. "Therefore the periodic and conservative fluctuation of Goodwin's dynamics is not dependent on irrational behaviors of players, but is rather a consequence of the conflicting elements proper of a pure capitalism" [4, p.64].

In a similar attempt, Mehrling [30] integrates GM with Lancaster [25]: capitalists are assumed to choose the level of investment to maximise the present value of their profits subject to the constraint that investment cannot be negative and cannot exceed capitalists' profit share. Instead, workers choose the rate of wage change to maximise the present value of their consumption stream, subject to an upper bound to their demands (i.e. the Phillips curve does not hold strictly):

(5) $\dot{w}/w \leq - g + rv$

The analysis of the economy under different institutional structures shows that, when both classes are unorganised, GM arises as an equilibrium of the game, with profits being completely reinvested and (5) binding.

However, neither attempt seems to provide satisfactory micro-foundations to GM. As shown in *Appendix 1*, a closer inspection of the equilibrium conditions shows that BCR's [4] conclusion is not warranted and Goodwin's parameters are basically assumed. As concerns Mehrling [30], first of all, constraint (5) remains unexplained. Secondly, even assuming, for the argument's sake, that the model effectively characterises the institutional setting in which Goodwin's parameters emerge as a rational choice of the two classes, Mehrling proves that if workers are unorganised, capitalists can improve by coordinating their actions (and vice versa). Therefore both classes have a strong incentive to abandon the atomistic behaviour and the *institutional structure* yielding GM is not robust.

Given the difficulty of providing sound micro-foundations to GM, it is not possible to argue that structural instability is not relevant because agents *choose* Goodwin's parameters, and the methodological issue regarding the opportunity to adopt a structurally unstable model cannot be avoided.

Section 2. Is structural instability enough to reject Goodwin's model?

In the literature on GM, structural instability has been considered a major shortcoming of the model.[12] Several extensions of GM have been explicitly aimed at yielding a structurally stable limit cycle: Medio [29] has introduced effective demand issues while assuming firms to adopt mark-up pricing, given money wage bargaining; Chiarella [5] has introduced a time lag in the Phillips curve, due to sluggish wage adjustment; Farkas and Kotsis [11], too, have modified the Phillips curve, introducing time lags and assuming logistic saturation, instead of

[12] With few exceptions, such as [6, 44]. According to Velupillai [41], "it must be remarked that the somewhat excessive concentration on the lack of structural stability ... seems to be highly misplaced." [41, p.81] "I am myself guilty of attaching a great deal of importance to this property. I am, now, less convinced about the importance and necessity of structural stability" [ibid., fn. 2].

exponential growth, at a zero real wage; Sportelli [37] has adopted an investment function including profit expectations.[13]

However, if, as shown by Vercelli [45], structural instability is not sufficient to reject GM *a priori*, it might be argued that GM actually represents a more satisfactory formalisation of distributive conflict than structurally stable models yielding limit cycles, from the point of view of both (a) the main characteristics of the theory that the model is meant to formalise; and (b) the basic features of the phenomenon investigated.

As concerns (a), models yielding limit cycles imply an extremely *reductionist* interpretation of Marx's theory, due to "the excessive determinism which derives from the structurally stable form." [49, p.185] For instance, in his seminal paper on LV systems [33], Samuelson rejects the structurally unstable specification of LV and, by setting up what might be considered the prototype of the models yielding limit cycles, he obtains "*a fundamental cyclical mechanism which could operate almost independently of the institutional environment*. ... If one posits two permanent features of economic life - exploitation and increasing returns - one can deduce a permanent business cycle (and moreover, one with a unique amplitude of fluctuations as well as period)" [33, p.473, italics added].

Instead, in a dialectic perspective, while the conflict between classes is inherent to capitalist societies, the structure of this contradiction tends to change over time, due to endogenous forces that tend to modify the balance of power between classes and the structure of the bargaining process [28, pp. 596-7]. The

[13] See also Van der Ploeg [39], Flaschel [13], Skott [35], Flaschel and Groh [14].

structural instability of GM might be interpreted as reflecting these forces, while structurally stable models imply a much narrower interpretation of Marx's theory.

As regards (b), the empirical evidence does not show the "fundamental cyclical mechanism" that would be implied by a limit cycle. In his evaluation of the U.S. post-war (u, v) plot, Solow points out two different phenomena: "a suggestion of predominantly clockwise motion, but in three separate episodes" [36, p.39]; and "a bare hint of a single large long-period clockwise sweep" [36, p.40]. Flaschel and Groh [14] and Harvie [21] find a remarkably similar pattern in the post-war (u, v) diagrams of, respectively, 8 and 10, OECD countries, with shorter (10-15 years) clockwise cycles around an incomplete long-period (50-70 years) clockwise motion. Hence, if anything, the "fundamental cyclical mechanism" represented by a limit cycle should describe the *long run* motions. Indeed, authors such as Farkas and Kotsis [11] or Flaschel and Groh [14], who have adopted a structurally stable generalisation of GM in their empirical work, have opted for an "approach which attempts to argue for a large phase length and amplitude of this growth cycle" [14, p.295].[14] From this perspective, the analysis should focus on a long run conflict between workers and capitalists, influenced by secular forces such that any shorter-run factors are averaged out. However, there are several reasons why this interpretation is not entirely convincing.

Firstly, it seems difficult to claim that there exists a long run empirical regularity to be explained in the first place. In several countries the long run motion is definitely not well shaped. In any case, "one cycle is not a periodic

[14] "Under the modern post-war conditions the large changes in income distribution among capital and labor which the [capitalistic] reproduction process calls for will use up an amount of time which can be drastically higher than the 8-10 years that Solow calculated" [14, p.294].

motion" [36, p.40] and much less so the "three-quarter cycle" [21, p.357] that emerges from the plots of the countries considered. In order to support a long run interpretation, one would need much longer time series than those considered in [11, 14, 21]. However, when longer time series are analysed, the existence of *stable* long run cycles is questionable, as shown forcefully in the U.K. 1855-1965 data by the "steady shift to the right (rising u)" [8, p.259] of the uv-trajectories. Furthermore, a long run interpretation seems inconsistent with Marx's theory of the *industrial reserve army* that these models are meant to formalise, since Marx explicitly considered cycles definitely shorter than 50-70 years (e.g. "a decennial cycle" [28, p.593 and *passim*]).

Therefore, it is legitimate to conclude that a structurally stable model yielding a limit cycle is not necessarily more appropriate to formalise distributive conflict. On the contrary, arguably, a less *reductionist* interpretation of distributive conflict is more in line with Marx's theory and is necessary to interpret the data since, "given that history does not repeat itself, the formal business cycle theory cannot contain all the truth." [17, p.18] However, in order to support this argument, it is necessary to provide an interpretation of GM consistent with the available empirical evidence.

Section 3. Goodwin's model reconsidered.

From a mathematical point of view, the LV system may be interpreted as the nonlinear analogue of the *frictionless oscillator*: "every physicist knows that this simple harmonic motion is best for explaining what oscillation is and how, why and when it arises. He is also aware that it is never found in practice." [18, p.68] Similarly, Goodwin's "starkly schematized and hence quite unrealistic" [15, p.54] model offers a stylised portrait *in vitro* of the basic mechanism underlying the interaction between classes. It models "the most essential dynamic aspects of capitalism. … [In order] to show the logic and plausibility of a type of behaviour and of its analysis, it is essential to get it clearly and simply stated" [16, p.443].

This is not inconsistent with the fact that, *empirically*, the mechanism described by GM may be subject to perturbations, as suggested by the available empirical evidence, and in particular by Desai's [8] and Harvie's [21] econometric analysis that leads to a rejection of the zero restrictions of GM.[15] As noted by Solow, "the model does capture something real. At least I can think of episodes that seem to conform to that [cyclical] pattern. I think that is all that can be said and all that need be said about a small model. It is no contradiction to say that the everyday business cycle marches to a different drumbeat. ... That is why, *methodologically speaking, small is beautiful*" [36, pp.38-9, italics added]. Thus, GM should be interpreted as a theoretical model to *understand* distributive conflict rather than as a formal *description* either of a long run (50-70 years) cycle[16] or of a short-run persistent fluctuation in income shares and employment.

However, given the proposed interpretation of the model, it is possible to suggest an alternative explanation of the *uv*-plots, based on the idea that

[15] In order to test the zero restrictions of GM, Desai [8] embeds it into a more general model with money wage bargaining and adaptive expectations. In the U.K. data 1855 – 1965, the assumption of constancy of the capital-output ratio is always rejected, while the results on money illusion and adaptive expectations are less clear-cut. However, Desai [8] concludes that the economy is more likely to be dynamically stable than to exhibit closed orbits. Applying the same method, Harvie [21] extends this conclusion to ten OECD countries for the period 1951–1991.

[16] In the empirical literature, the long run interpretation of GM is endorsed, e.g., by Harvie [21].

Goodwin's (and Marx's) cycles are the shorter run (10-15 years) cycles appearing around the long run motion. The shorter-run cycles tend to become visible in a less perturbed environment, while "structural change … over the period is causing their position to shift." [21, p.357] This interpretation is consistent with the rejection of the zero restrictions of LV, and thus with the fact that structural instability is also *empirically* relevant [7, 21]. Moreover, as noted by Atkinson [3], the period T of Goodwin's cycles is determined by the model's parameters, and the values of T derived in the empirical literature are consistent with the suggested interpretation of GM.[17] Actually, if the long run motion is interpreted as the product of structural change, this might also explain the observed irregularity of shorter-run cycles, since a shift of the *centre* implies a change in T. Finally, given the arguments discussed in *section 2*, this explanation of the empirical evidence seems more satisfactory than the long run interpretation; while, on the other hand, the plots show no evidence of short run *stable* fluctuations.

From this perspective, unlike structurally stable models, the interest for GM lies *both* in the description *in vitro* of the basic forces underlying distributive conflict *and* in the fragility of the structure of this interaction, which allows for a less "deterministic" interpretation of the cycles. Thus, the *frictionless oscillator* describes a periodic motion in an ideal environment, but it might be argued that its structural instability also indirectly helps to explain frictions at a very small computational cost. Similarly, GM provides a stylised description of distributive

[17] With a linear Phillips curve [21], $T = (4p)/[(a + g)(1 - (a + b)s)]^{1/2}$. Atkinson [3] computes a T between 9.8 and 21.9 and van der Ploeg [39] between 10 and 16; Solow considers " a time scale considerably longer than the ordinary business cycle," [36, p.40] namely 8 or 10 years.

conflict *in vitro*, but given its structural instability, it can also be interpreted as a convenient tool to understand structural changes.

From a theoretical viewpoint, structural instability makes GM particularly suitable for an analysis of the changes in the dynamic features of the model, and thus in the structure of the interaction between classes, arising from perturbations of the assumptions.[18] The theoretical importance of this issue is explicitly recognised by van der Ploeg, who analyses "Structural perturbations of the predator-prey model of the class struggle." [39, sect. 5] However, a significant part of the literature on GM can be interpreted as analysing the structural changes induced by perturbations of the model's assumptions. These contributions focus in particular on the existence of cycles, however the potential structural changes are richer and more interesting both in theoretical and in analytical terms. Mathematically, they involve issues like the existence and multiplicity of the equilibria and their dynamic properties. From an economic viewpoint, the perturbed models describe different economic and institutional environments with extremely different positive and normative implications, as concerns the outcome of distributive conflict and the possibilities for government intervention.[19]

Consider, for instance, the papers by Desai [7] and Desai and Shah [10], which analyse extensions of GM of gradually increasing generality. In a first

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[18] However, it is *not* true that GM "is not suitable for any extension due to its structural instability" [29, p.33]: Izzo's [23] and van der Ploeg's [38] extensions preserve the dynamics of GM.

[19] For instance, van der Ploeg notes that "fear of redundancies, money illusion and adaptive expectations, under both the cost-push and the quantity theory of inflation, tend to dampen class conflict. On the other hand, the hoarding of labour and the effect of the expected return on capital on the capital-output ratio give rise to exploding cycles of conflict. … When one allows for

extension, Desai [7] shows that the introduction of money wage bargaining and inflation has a stabilising effect and the system displays dampened cycles around the equilibrium. Moreover, "the presence of inflation means that to get an equivalent rise in real wage rate in the money wage bargaining situation, a higher level of employment (a higher bargaining strength) is necessary." [7, p.533] Hence, Desai argues that "inflation is … a way of redistributing income away from labor." [7, p.533] Actually, inflation plays this role only *potentially*. Although the role of the parameters affecting the price dynamics (and the position of the phase curves) suggests that in general inflation might affect workers' bargaining strength adversely, in this extension if the rate of inflation is positive, in the steady state v is higher than in GM, while u is the same, *ceteris paribus*. Hence, from the point of view of the equilibrium, the stabilising effect of inflation does not seem negative for workers, unless a positive value is attached to conflict *per se*, regardless of the distributive outcome.[20]

If, in addition to inflation and money wage bargaining, a flexible capital/output ratio is introduced, as shown in *Appendix 2* several major structural changes occur, in addition to those analysed in [7]. Firstly, although Desai [7] focuses only one equilibrium (empirically the most likely, with high values of v and u), *multiple equilibria* emerge, up to potentially four non-trivial equilibria. Secondly, these equilibria have different stability properties, while in general the economy shows no permanent cycles and fluctuations tend to disappear even as

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substitution between the factors of production, or for the optimal choice of the direction of technical progress, conflict is completely eliminated" [39, p.277].

[20] Another structural change, not discussed in [7], regards the fact that too high a mark-up can be inconsistent with the *existence* of an equilibrium for admissible values of u and v.

transitory phenomena. Finally, capitalists "control" the main parameters determining the position of the two phase curves (especially those affecting the price dynamics) so that, in a sense, they determine the steady state values of the two variables. Thus, in this extension inflation displays its negative effects on the workers' bargaining strength due to money illusion *and* a less rigid assumption on technology. Moreover, unlike GM, the existence of multiple equilibria leaves room for government intervention in the bargaining process in order to prevent the economy from being trapped in a low-values equilibrium.[21]

Finally, if one allows for the choice of the direction of technical change, as in [10], the steady state is unique and locally asymptotically stable. "Unlike in GM, capitalists learn from history in this model. They invest all their profits but they also choose the new technology that will be cost minimising ... any small deviation from equilibrium will allow the capitalists to claw back gains made by workers. ... At the heart of this result lies the *importance of allowing an extra degree of freedom to one party in a model of class struggle*" [10, pp.1008-9, italics added]. Apart from the absence of persistent cycles, another crucial difference with respect to GM is that the steady state values of u and v are entirely determined, in the last instance, by the capitalists' cost minimisation decisions. An extra degree of freedom yields a change *both* in the stability properties of the equilibrium *and* in its value.

The brief discussion of these two papers does not exhaust the range of possible structural changes induced by perturbations of the assumptions of GM.

[21] Similar structural changes obtain if inflationary adaptive expectations are added, as in Desai [7, pp.539-543].

However, it provides an illustration of the economic and mathematical issues that can be analysed by perturbing the model, and therefore it suggests a line for further research, especially concerning a thorough analysis of the literature on GM, based on the interpretation of the model proposed in this paper.

Section 4. Conclusions.

In this paper, the methodological prescription to reject a structurally unstable model is critically analysed by considering the specific case of GM [15, 16]. The inherent ambiguities in the formal definition of structural instability and the difficulties of proving the structural instability of GM are highlighted. Moreover, based on Vercelli's [45] arguments, showing that structural instability is not a sufficient reason to reject *a priori* a dynamic system, GM is compared with structurally stable models of distributive conflict, taking into account both the empirical evidence and the main characteristics of the theory that underlies the model. It is argued that from both points of view, GM represents a more satisfactory formalisation of Marx's theory of distributive conflict.

An interpretation of GM based on its structural instability is proposed. It is argued that Goodwin's starkly schematised model isolates *in vitro* the main forces underlying distributive conflict, and it might be inappropriate to use it *as such* to describe the data pattern in a perturbed environment. However, given its structural instability, GM may also be interpreted as a convenient tool to understand structural changes and an explanation of the empirical evidence based on this interpretation of GM is provided. It is suggested that it is more appropriate to identify Goodwin's cycles with the shorter-run cycles visible in the *uv*-plots around a long run motion, while the latter should be taken as the product of structural change.

Given the methodological approach proposed and the suggested interpretation of GM, it seems evident that additional empirical research is needed. Firstly, Vercelli's [43] methodology, based on the distinction between movements around the *centre* and shifts of the equilibrium, seems particularly suitable for a *qualitative* evaluation of GM. Actually, Vercelli's [43] results for four OECD countries are encouraging and it might be opportune to extend his analysis to a larger set of countries for a longer time period.

Secondly, at a *quantitative* level, if the suggested interpretation is correct, a straightforward least squares estimation of GM would not yield plausible estimates, unless structural changes are accounted for. For instance, as argued in Veneziani [42], this might explain the rather implausible estimates obtained by Harvie [21]. As suggested by Desai [9], a full stochastic specification of GM might be opportune. Although this is a rather difficult task (that might lead "into the abstruse area of Ito processes" [8, p.256]), "it may enable us to explore a richer set of [dynamic] outcomes" [9, p.5] and it may give the opportunity to take into account structural changes.

APPENDIX

A1. Mathematical analysis of Balducci, Candela and Ricci [4].

Let $1 - v \cong -\log v \equiv x_1$ and $1 - u \cong -\log u \equiv x_2$, so that $\dot{x}_1 = -\dot{v}/v$ and $\dot{x}_2 = -\dot{u}/u$. BCR allow a fraction $1 - s$ of profits not to be reinvested. Thus, retaining the notation used so far

$$(1.1)\ \dot{x}_1 = -sx_2/s + (a + b)$$

$$(1.2)\ \dot{x}_2 = (g + a) + rx_1$$

where $0 < s \leq 1$. The equilibrium of the system is given by

$$(1.3)\ x_1^* = -(g + a)/r$$

$$(1.4)\ x_2^* = s(a + b)/s$$

(1.1) and (1.2) represent the dynamic state equations of the differential game. The strategic variable of player 2, capitalists, is the percentage of profits to be reinvested: $u_2 = s$; while the strategic variable of player 1 workers is the intensity $u_1 = r$ of claimed wages increase, where $u_1 > 0$ and $0 < u_2 \leq 1$.

Given diminishing marginal utility of consumption, BCR assume the objective functions of the two players to depend on the deviations between actual and desired values of the *state* variables. In particular, let x_j^i, $i, j = 1, 2$ be the steady state value of the j - th variable desired by the i - th player (both players care about both variables). Then, from (1.3) and (1.4)

$x_1^i = (S' - a)/u_1^i$

$x_2^i = s(a + b)/u_2^i$

Since $x_j - x_j^i = x_j\{(u_j^i - u_j)/u_j^i\} \cong x_j(u_j^i - u_j)$, $i, j = 1, 2$, the loss function are

$$J^i = ½ \int_0^{\infty} [x_1(u_1^i - u_1)^2 + x_2(u_2^i - u_2)^2] \, dt$$

Thus, agent i's Hamiltonian, $H^i \equiv H^i(x, p_1^i, p_2^i)$ can be written as

$$H^i = ½ [x_1(u_1^i - u_1)^2 + x_2(u_2^i - u_2)^2] + p_1^i [- u_2 x_2/s + (a + b)] + p_2^i [u_1 x_1 - (S' - a)]$$

The first order conditions for the two classes are:

(i) $\partial H^1/\partial u_1 = - x_1(u_1^1 - u_1) + p_2^1 x_1$

(ii) $\partial H^2/\partial u_2 = - x_2(u_2^2 - u_2) - (1/s)\, p_1^2 x_2$

(iii) $\partial H^1/\partial x_1 = ½ (u_1^1 - u_1)^2 + p_2^1 u_1 = - \dot{p}_1^1$

(iv) $\partial H^1/\partial x_2 = ½ (u_2^1 - u_2)^2 - (1/s)\, p_1^1 u_2 = - \dot{p}_2^1$

(v) $\partial H^2/\partial x_1 = ½ (u_1^2 - u_1)^2 + p_2^2 u_1 = - \dot{p}_1^2$

(vi) $\partial H^2/\partial x_2 = ½ (u_2^2 - u_2)^2 - (1/s)\, p_1^2 u_2 = - \dot{p}_2^2$

From (i) and (ii), the Nash solution (u_1*, u_2*) for this problem is

(vii) $u_1^* = u_1^1 - p_2^1$

($viii$) $u_2^* = u_2^2 + p_1^2/s$

Given that the optimal Hamiltonians are affine functions of x_1 and x_2, each player has a value function, $V^i(x_1, x_2) = a_1^i x_1 + a_2^i x_2 + g^i$, satisfying the Hamilton-Bellman partial differential equation $\partial V/\partial t = -H^i(x_1, x_2)$, with $p_j^i = \partial V/\partial x_j = a_j^i$.

Hence substitute a_j^i for p_j^i in (iii)-($viii$). Define an *Extremal Steady State* as a stationary solution of (iii)-(vi) and (1.1)-(1.2). From (vii) and ($viii$) it is clear that if in an *ESS* $a_2^1 = a_1^2 = 0$, as claimed by BCR, then players *choose* $u_1^* = u_1^1$ and $u_2^* = u_2^2$, i.e. the parameters yielding GM, which is thus "proved to coincide with the *ESS* of a non-cooperative Nash game" [4, p.64]. However, from (iii')-(vi'):

(iii'') $a_2^1 = 2u_1^1$

(vi'') $a_1^2 = - 2 s u_2^2$

(iv'') $a_1^1 = s\ [u_2^1 + u_2^2]^2/(2u_2^2)$

(v'') $a_2^2 = [u_1^2 + u_1^1]^2/(2u_1^1)$

By using (iii'') and (vi'') into (vii') and ($viii$'): $u_1^* = - u_1^1$ and $u_2^* = - u_2^2$. Hence, given $u_1^1, u_2^2 > 0$, the only possible way to have $u_1^* = u_1^1$ and $u_2^* = u_2^2$ is to impose $a_2^1 = a_1^2 = 0$ as an *a priori* restriction, so that $J^i = ½ \int x_i(u_i^i - u_i)^2\, dt$, i = 1, 2. Trivially, the J^i's have a unique minima corresponding to $u_i^i = u_i$, i = 1, 2. Therefore, the parameters are not derived from the general problem; the latter has to be modified *ad hoc* in order to get the desired result.

A2. Mathematical analysis of structural changes in Desai [7].

A2.1. Inflation [7, pp.531-36].

Assume money wage, m, bargaining and that "in equilibrium price [p] equals a mark-up times the unit labour cost of output in money terms but observed price adjusts to this in an exponentially distributed lag form" [7, pp.531-2].

A4') $\dot{m}/m = - g_1 + r_1 v$

A5) $\dot{p}/p = l_1(\log m - \log a - \log p + \log p)$

Given A4', A5 and A3 and $u \equiv m/pa$, $\dot{p}/p = 0 \Rightarrow p = mp/a$:

$$(2.1)\quad \dot{u} = - [(g_1 + a) + l_1 \log p]\, u + r_1 v u - l_1 u \log u$$

The singular point of the system (1)-(2.1) is given by $u^{**} = u^* = 1 - s(a + b)$ and $v^{**} = [(g_1 + a) + l_1 \log p + l_1 \log u]/r_1$.

Hence: (1) *ceteris paribus*, while $u^{**} = u^*$, if $\dot{p}/p > 0$ then $v^{**} > v^*$; (2) from v^{**} it emerges that a high value of p can be inconsistent with the *existence* of an equilibrium in the unit box. Finally, it is not difficult to show that inflation has a stabilising effect and the system shows damped cyclical movements around the (unique) equilibrium [7, p.536].

A2.2. Money illusion and flexible capital/output ratio [7, pp.536-9].

If money illusion is introduced, A4' becomes

A4") $\dot{m}/m = -g_1 + r_1 v + h_1 (\dot{p}/p)$, $0 \leq h_1 \leq 1$

Given A4", the equation of motion for u becomes

$$\text{(2.2)} \quad \dot{u} = -[(g_1 + a) + (1 - h_1) l_1 \log p]u + r_1 uv + (h_1 - 1) l_1 u \log u$$

As a second step, relax the assumption about technology. Assume

A1') $s = s^* v^{-m} \Rightarrow \dot{s}/s = -m(\dot{v}/v)$

where s^* is the desired value of the capital/output ratio. Given A1',

$$\text{(2.3)} \quad \dot{v} = v^{m+1}(1 - u)/(1 - m)s^* - (a + b)v/(1 - m)$$

The equilibrium loci are derived setting $\dot{v} = \dot{u} = 0$:

$$\text{(2.4)} \quad v^{**} = [(g_1 + a) + (1 - h_1) l_1 \log p + (1 - h_1) l_1 \log u]/r_1$$

$$\text{(2.5)} \quad u^{***} = 1 - s^*(a + b)v^{-m}$$

From (2.4), $v^{**} = 0 \Rightarrow u = (1/p) \exp [(g_1 + a)/(h_1 - 1) l_1]$ and, from (2.5), $u^{***} = 0 \Rightarrow v = [s^*(a + b)]^{1/m}$. Given $p > 1$, $h_1 < 1$ and $[s^*(a + b)] < 0 \Rightarrow 0 < u, v < 1$. From (2.5), $v = 1 \Rightarrow u^{***} = 1 - s^*(a + b)$, and from (2.4), $u = 1 \Rightarrow v^{**} = [(g_1 + a) + (1 - h_1) l_1 \log p]/r_1$. Moreover, as noted by Desai [7, p.537], $dv^{**}/du > 0$, $d^2v^{**}/du^2 < 0$, while after inverting (2.5) $dv/du^{***} > 0$ and $d^2v/d(u^{***})^2 > 0$.

Hence, both u^{***} and v^{**} intersect one axis in $I = [0, 1] \times [0, 1]$, but it is not possible to say whether they intersect each other (once or twice) in I, as their shapes would suggest. A high p and a low m make an intersection more likely, but if p is too high the second intersection could happen outside I; on the other hand, a low h tends to make the intersections more likely. *Assume*, as in Desai [7, fig.3], that they intersect twice in I. Let $y \equiv \log u$ and $x \equiv \log v$

$$(2.6) \quad \dot{y} = -[(g_1 + a) + (1 - h_1) l_1 \log p] + (h_1 - 1) l_1 y_1 + r_1 e^x$$

$$(2.7) \quad \dot{x} = -(a + b)/(1 - m) + [1/s^*(1 - m)]\,(1 - e^y)\, e^{mx}$$

so that $\dot{x} = 0 \Rightarrow m[1/s^*(1 - m)]\,(1 - e^{y^*})\, e^{mx^*} = m\,(a + b)/(1 - m)$.

Linearising (2.6)-(2.7) around the equilibrium, the Jacobian J_1 obtains

$-(1 - h_1) l_1$	$r_1 e^{x^*}$
$-[1/s^*(1 - m)]\, e^{y^*} e^{mx^*}$	$m[1/s^*(1 - m)]\,(1 - e^{y^*})\, e^{mx^*}$

$\mathrm{tr}J_1 = -l_1(1 - h_1) + m\,(a + b)/(1 - m)$

$\det J_1 = 1/s^*(1 - m)\,[-l_1(1 - h_1) m\, s^*(a + b) + r_2 v^{*m+1} u^*]$

Hence, if the two equilibria are sufficiently close to the boundary of I, the following is likely to happen: given $h_1 < 1$, (a) $\mathrm{tr}J_1 < 0$ if $m > 1$, or if $m < 1$ but $(a + b)$ is sufficiently small; (b) if $m < 1$, $\det J_1 > 0$ ($\det J_1 < 0$) in the equilibrium with high (low) values of v^{**} and u^{***}. Instead if $m > 1$, $\det J_1 < 0$ ($\det J_1 > 0$) in the equilibrium with high (low) values of of v^{**} and u^{***}.[22] Moreover, since the discriminant of the characteristic equation is $D = ((-\mathrm{tr}J_1)^2 - 4 \det J_1)$, in proximity of the two unstable equilibria ($\det J_1 < 0$) there are no cycles, and since $D = [(h_1 - 1) l_1 - m(a + b)/(1 - m)]^2 - 4 r_1 [v^{m+1} u/s^*(1 - m)]$, if $m > 1$ there are no cycles around the low-values stable equilibrium either.

[22] Since $d\dot{v}/du = -v^{m+1} / s^*(1 - m)$ then if $u > u^{***}$, (a) if $m < 1$, $\dot{v} < 0$; (b) if $m > 1$, $\dot{v} > 0$.

Apart from the two equilibria corresponding to the intersections of the phase curves $\dot{v}$ = 0 and $\dot{u}$ = 0 there are other equilibria of the system: (1) the equilibrium corresponding to $v = 0$, $u = 0$ that is unstable, in both $m > 1$ and $m < 1$ (in the latter case it is, locally, a saddle), (2) the intersections of the u*** and v** curves with, respectively, the ordinate and the abscissa axes.

Consider the system formed by (2.2) and (2.3). Let (u', v') denote a generic point in the plane. The Jacobian, J_2, of the system around (u', v') is

$-(g_1+\alpha)-(1-h_1)l_1\log p-(1-h_1)l_1(1+\log u')+r_1v'$	$r_1 u'$
$-[1/s*(1-m)](v')^{m+1}$	a

where $a = (m+1)\dfrac{(1-u')v'^m}{s*(1-m)} - \dfrac{(\alpha+\beta)}{(1-m)}$

$\text{tr}J_2 = -[(g_1+\alpha)+(1-h_1)l_1\log p]-(1-h_1)l_1(1+\log u')+r_1v'+a$

$\det J_2 = \{-[(g_1+\alpha)+(1-h_1)l_1\log p]-(1-h_1)l_1(1+\log u')+r_1 v'\}a+[1/s*(1-m)](v')^{m+1}r_1 u'$

Consider first the equilibrium with $u' = 0$ and $v' = [s*(\alpha+\beta)]^{1/m}$: $r_1 u = 0$ and $-(\alpha+\beta)/(1-m)+[1/s*(1-m)]v^m = 0$. Notice that $u \to 0 \Rightarrow \text{tr}J_2 \to \infty$ in both cases. Instead, $m < 1 \Rightarrow \det J_2 \to \infty$ and $m > 1 \Rightarrow \det J_2 \to -\infty$, yielding a locally unstable equilibrium in the former case and a saddle in the latter.

Consider next the equilibrium with $v = 0$ and $u' = (1/p)\exp[(g_1+\alpha)/l_1(h_1-1)]$, and with $[1/s*(1-m)](m+1)(1-u')(v')^m = -[1/s*(1-m)](v')^{m+1} = 0$. In this case, $\text{tr}J_2 = -(1-h_1)l_1-(\alpha+\beta)/(1-m)$ and $\det J_2 = (1-h_1)l_1(\alpha+\beta)/(1-m)$, therefore $m < 1 \Rightarrow \text{tr}J_2 < 0$ and $\det J_2 > 0$ and the equilibrium is (locally) stable, while $m > 1 \Rightarrow \text{tr}J_2 > 0$ or $\text{tr}J_2 < 0$ but $\det J_2 < 0$, and the equilibrium is a saddle.

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